

Physics 128 Discussion (Week 1)

Problem 1

Express the following quantities in natural unit (GeV)

(a) Electron mass $m = 9.11 \times 10^{-31}$ kg

$$m \rightarrow mc^2 \approx 0.511 \times 10^{-3} \text{ GeV} \quad (1)$$

(b) Wavelength of light $\lambda = 500$ nm

$$\lambda \rightarrow \frac{\lambda}{\hbar c} \approx 2.54 \times 10^9 \text{ GeV}^{-1} \quad (2)$$

(c) Time interval $t = 1$ ns

$$t \rightarrow \frac{t}{\hbar} \approx 1.52 \times 10^{15} \text{ GeV}^{-1} \quad (3)$$

(d) Temperature $T = 10^5$ K

$$T \rightarrow k_B T \approx 8.617 \times 10^{-9} \text{ GeV} \quad (4)$$

Problem 2

Express the following quantities in SI unit

(a) Particle with mass $m = 1$ GeV

$$m \rightarrow mc^2 = 1 \text{ GeV} \Rightarrow m \approx 1.783 \times 10^{-27} \text{ kg} \quad (5)$$

(b) Wavelength of light $\lambda = 0.5 \text{ GeV}^{-1}$

$$\lambda \rightarrow \frac{\lambda}{\hbar c} = 0.5 \text{ GeV}^{-1} \Rightarrow \lambda \approx 9.9 \times 10^{-2} \text{ fm} = 9.9 \times 10^{-17} \text{ m} \quad (6)$$

(c) Distance $x = 30 \text{ GeV}^{-1}$

$$x \rightarrow \frac{x}{\hbar c} = 30 \text{ GeV}^{-1} \Rightarrow x \approx 5.91 \text{ fm} = 5.91 \times 10^{-15} \text{ m} \quad (7)$$

(d) Temperature $T = 0.2 \text{ GeV}$

$$T \rightarrow k_B T = 0.2 \text{ GeV} \Rightarrow T \approx \frac{0.2 \text{ GeV}}{8.617 \times 10^{-5} \text{ eV/K}} \approx 2.32 \times 10^{12} \text{ K} \quad (8)$$

Problem 3

A particle with rest mass M splits from rest into two particles with masses m_1 and m_2 . In this case, find the four-momentum of each particle. We use natural units throughout this question.

Let us consider in the rest frame S of the original particle. Without loss of generality, you can set the generated particles to move along the z -axis

$$P^\mu = \begin{pmatrix} M \\ 0 \end{pmatrix}, \quad p_1^\mu = \begin{pmatrix} \sqrt{m_1^2 + p^2} \\ p \end{pmatrix}, \quad p_2^\mu = \begin{pmatrix} \sqrt{m_2^2 + p^2} \\ -p \end{pmatrix}. \quad (9)$$

Energy-momentum conservation reads

$$P^\mu = p_1^\mu + p_2^\mu. \quad (10)$$

Taking the Lorentz-squared norm of both sides, you get

$$P^2 = P^\mu P_\mu = M^2 \quad (11)$$

$$\begin{aligned} (p_1 + p_2)^2 &= (p_1^\mu + p_2^\mu)(p_{1\mu} + p_{2\mu}) \\ &= p_1^2 + p_2^2 + 2p_1^\mu p_{2\mu} \\ &= m_1^2 + m_2^2 + 2\sqrt{(m_1^2 + p^2)(m_2^2 + p^2)} + 2p^2. \end{aligned} \quad (12)$$

Therefore

$$\begin{aligned} m_1^2 + m_2^2 + 2\sqrt{(m_1^2 + p^2)(m_2^2 + p^2)} + 2p^2 &= M^2 \\ \Rightarrow 2\sqrt{(m_1^2 + p^2)(m_2^2 + p^2)} &= M^2 - (m_1^2 + m_2^2) - 2p^2 \\ \Rightarrow 4M^2 p^2 &= [M^2 - (m_1^2 + m_2^2)]^2 - (2m_1 m_2)^2 \\ p^2 &= \frac{[M^2 - (m_1 + m_2)^2][M^2 - (m_1 - m_2)^2]}{4M^2} \\ \Rightarrow p &= \frac{\sqrt{[M^2 - (m_1 + m_2)^2][M^2 - (m_1 - m_2)^2]}}{2M}. \end{aligned} \quad (13)$$

From this p , p_1^μ and p_2^μ are uniquely determined.

Problem 4

A massive particle of energy E moving in the x -direction collides with a particle of mass m at rest. In frame S , the four-momenta corresponding to the two particles are

$$p^\mu = \begin{pmatrix} E \\ p \end{pmatrix} \quad q^\mu = \begin{pmatrix} m \\ 0 \end{pmatrix} \quad (14)$$

We use natural units throughout this question.

(a) The quantity s is defined by $s \equiv (p + q)^\mu (p + q)_\mu$. Here, $(p + q)^\mu = p^\mu + q^\mu$. Find $\sqrt{s}$ in terms of E , p , and m .

$$\begin{aligned} s &\equiv (p + q)^2 \\ &= \left(\begin{pmatrix} E + m \\ p \end{pmatrix} \right)^2 \\ &= (E + m)^2 - p^2 \end{aligned} \quad (15)$$

$$\Rightarrow \sqrt{s} = \sqrt{(E + m)^2 - p^2} \quad (16)$$

(b) Frame S' moves at a velocity $+\beta\hat{x}$ in frame S . Find p'^μ and q'^μ in terms of E , p , m , β , and γ .

In general, the Lorentz transformation is given by

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \quad (17)$$

Then, the boosted momentum p'^μ, q'^μ are given by

$$\begin{aligned} p'^\mu &= \Lambda^\mu{}_\nu p^\nu \\ &= \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} E \\ p \end{pmatrix} \\ &= \begin{pmatrix} \gamma(E - \beta p) \\ \gamma(p - \beta E) \end{pmatrix} \end{aligned} \quad (18)$$

$$\begin{aligned} q'^\mu &= \Lambda^\mu{}_\nu q^\nu \\ &= \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} m \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \gamma m \\ \gamma\beta m \end{pmatrix} \end{aligned} \quad (19)$$

(c) What value of β is needed so that $\mathbf{p}' + \mathbf{q}' = 0$? If $\mathbf{p}' + \mathbf{q}' = 0$, then frame S' is called the *center-of-momentum frame*.

From (b), we have

$$p'^\mu = \begin{pmatrix} \gamma(E - \beta p) \\ \gamma(p - \beta E) \end{pmatrix}, \quad q'^\mu = \begin{pmatrix} \gamma m \\ \gamma\beta m \end{pmatrix} \quad (20)$$

Requiring $\mathbf{p}' + \mathbf{q}' = \mathbf{0}$, we get

$$\begin{aligned} \gamma[p - \beta(E + m)] &= 0 \\ \Rightarrow \beta &= \frac{p}{E + m}, \quad \gamma = \frac{1}{\sqrt{1 - \frac{p^2}{(E + m)^2}}} \end{aligned} \quad (21)$$

(d) Compute $\sqrt{s}$ directly in the center-of-momentum frame. Since s is a Lorentz scalar, your answer should be the same as in part (a).

$$\begin{aligned}
 s' &\equiv (p' + q')^2 \\
 &= \left(\gamma \begin{pmatrix} (E + m) - \frac{p^2}{E + m} \\ 0 \end{pmatrix} \right)^2 \\
 &= \gamma^2 \left[(E + m) - \frac{p^2}{E + m} \right]^2 \\
 &= (E + m)^2 - p^2 \\
 &= s
 \end{aligned} \tag{22}$$

$$\Rightarrow \sqrt{s'} = \sqrt{s} = \sqrt{(E + m)^2 - p^2} \tag{23}$$